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The Bajnok-Janik formula and wrapping corrections

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Abstract: We write down the simplified TBA equations of the $AdS_5 \times S^5$ string σ-model for minimal energy twist-two operators in the $sl(2)$ sector of the model. By using the linearized version of these TBA equations it is shown that the wrapping corrected Bethe equations for these states are identical, up to $O(g^8)$, to the Bethe equations calculated in the generalized Lüscher approach (Bajnok-Janik formula). Applications of the Bajnok-Janik formula to relativistic integrable models, the nonlinear $\mathrm{O}(n)$ sigma models for $n = 2, 3, 4$ and the $\mathrm{SU}(n)$ principal sigma models, are also discussed.

Contents

1. Introduction

One of the most important problems in testing the AdS/CFT correspondence [1] is to understand the finite size spectrum of the $AdS_5 \times S^5$ superstring. For large volumes the asymptotic Bethe Ansatz (ABA) describes the spectrum of the model [2]. It takes into account all power like corrections in the size, but neglects the exponentially small wrapping corrections [3].

In [4, 5] it was shown that the leading order wrapping corrections can also be expressed by the infinite volume scattering data through the generalized Lüscher formulae [6]. In [4] the 4-loop anomalous dimension of the Konishi operator was obtained by means of the generalized Lüscher formulae in perfect agreement with direct field theoretic computations [7, 8]. Subsequently wrapping interactions computed from Lüscher corrections were found to be crucial for the agreement of some structural properties of twist-two operators [9] with LO and NLO BFKL expectations [10, 11].

More recently [12] the 5-loop wrapping correction to the anomalous dimension of the Konishi operator was also computed from the generalized Lüscher approach accounting for the expected nontrivial transcendentality structure of the anomalous dimension. Later the 5-loop result has been extended to the class of twist two operators as well [13]. After analytic continuation to negative values of the spin this gave nontrivial agreement with the predictions of the BFKL equations [10].

Although the generalized Lüscher approach was invented for the purpose of computing the wrapping corrections in the AdS/CFT context, it has more general validity. In particular, it is also valid in relativistic integrable models, like the Sine-Gordon model, nonlinear

σ-models and other related models. Based on the TBA/NLIE description of these models, we have proven the validity of the Bajnok-Janik approach to generalized Lüscher corrections for them. In appendix C we summarize the results for the O(2), O(3) and O(4) nonlinear σ-models and the SU(n) principal model.

After the discovery of integrability of the string worldsheet theory the mirror Thermodynamic Bethe Ansatz (TBA) technique was used [3, 14] to determine the exact spectrum of string theory (including the exponentially small Lüscher corrections). The TBA equations of AdS/CFT were derived first for the ground state [15, 16, 17, 18, 19] and then using an analytic continuation trick [20] excited states TBA equations were conjectured for the excitations of the $sl(2)$ sector of the theory [19, 21, 22]. Since the final form of the TBA equations is still a conjecture it is important to test them carefully. In the strong coupling limit it was shown [23, 24] that the TBA equations reproduce correctly the 1-loop string energies in the quasi-classical limit. On the other hand in the opposite weak coupling limit it is of fundamental importance to see that the TBA equations are consistent with the generalized Lüscher formulae. This can be tested by studying the small g expansion[1] of both the Lüscher formulae and the TBA equations.

In the TBA context the string energies are given by the formula:

$$E = J + \sum_{i=1}^{N} \mathcal{E}(p_i) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \frac{d\widetilde{p}^Q}{du} \log(1 + Y_Q)\,, \tag{1.1}$$

where N is the number of particles, J is the angular momentum carried by the string rotating around the equator of S^5, $\widetilde{p}^Q$ is the mirror momentum and the functions Y_Q are the unknown functions (Y-functions) associated to the mirror Q-particles, furthermore

$$\mathcal{E}(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}} \tag{1.2}$$

is the dispersion relation of the string theory particles.

The wrapping corrections to string energies have two sources: the momenta of the particles change due to wrapping corrections to the ABA and there is a contribution proportional to the asymptotic form of the Y_Q functions. Since the asymptotic form of the Y_Q functions are built into the TBA equations by construction, only the consistency of the wrapping corrected forms of the ABA obtained from the TBA and the generalized Lüscher approach has to be verified. This comparison has been done first numerically [25] then analytically [26] for the Konishi state in leading order[2] in g. In this paper we extend this comparison to the familiy of minimal energy twist-two operators $\mathrm{Tr}(D^N Z^2) + \dots$ of the $sl(2)$ sector of the theory.

According to the generalized Lüscher approach the leading order wrapping corrected Bethe equations for the twist-two ($J = 2$) operators take the form:

$$\pi(2n_k + 1) \,=\, J\, p_k + i \sum_{j=1}^{N} \log S_{\mathfrak{sl}(2)}^{1_* 1_*}(u_j, u_k) + \delta\mathcal{R}_k^{(\mathrm{BJ})} \,+ O(g^9), \tag{1.3}$$

[1] Here g is the coupling constant related to the 't Hooft coupling λ through $\lambda = 4\pi^2 g^2$.

[2] I.e. at the order of g^8.

where n_k is the integer quantum number characterizing the corresponding rapidity u_k and $\delta\mathcal{R}_k^{(\mathrm{BJ})}$ is the order g^8 wrapping correction to the ABA, obtained from the Bajnok-Janik formula [4]:

$$\delta\mathcal{R}_k^{(\mathrm{BJ})} = \frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{\partial}{\partial u_k}\,Y_Q^{asympt}(u)\big|_{\{u_j\}=\{u_j^o\}}\,. \tag{1.4}$$

Here $Y_Q^{asympt}(u)$ is of order g^8 and is a function of u and the particle rapidities u_j. After the differentiation the particle rapidities are taken at the solution of the ABA, u_j^o. We note that the simple formula (1.4) is valid only for the leading order wrapping corrections of the Bethe equations. At higher orders in g the corrections can no longer be expressed by the derivative of the Y_Q functions with respect to the magnon rapidities u_k. The more general nonrelativistic formula can be found in [4].

In this paper we will prove that by expanding the TBA equations we can exactly reproduce the formulae (1.3),(1.4) for the g^8 order wrapping corrections of the Bethe equations for the twist-two operators. The proof is a generalization of that used for the case of the Konishi operator in [26].

The paper is organized as follows. In section 2 we write down the TBA equations for the twist-two operators. In section 3 we linearize them around the asymptotic solution to describe the wrapping effects. In section 4 we present our results and the paper is finished with some conclusions. The technical details of the calculation are discussed in appendix A and we give a derivation of the leading order Bajnok-Janik formula (1.4) in appendix B. Appendix C contains the Bajnok-Janik formula for some relativistic integrable models.

2. Simplified TBA equations for the twist-two operators

Excited state TBA equations were proposed for certain classes of states in the $sl(2)$ sector in [19, 22]. The TBA equations of ref. [19] are valid for states where only the singularities associated to the function Y_1 have to be taken into account while [22] contains the detailed analysis of the two-particle states. Since our states of interest are not discussed[3] in the above papers here we write down the TBA equations for the twist-two operators valid for small values of the coupling. We proceed in the spirit of [22], namely it is assumed that the TBA equations are formally the same for the ground state and excited states provided the integration contours are defined properly. The form of the equations becomes different when the integration contours are deformed back to the real line of the mirror theory picking up the contribution of the poles from the convolution terms. The necessary singularity structure of the Y-functions can be read off from their asymptotic form.

The twist-two operators are given by $N/2$ pair[4] of real rapidities $\{u_j, -u_j\}$ and the asymptotic form of the Y-functions associated to these states can be obtained by using the results of [27]. Inspecting the analyticity properties of the Y-functions the TBA equations for the twist-two operators can be written down and take the following form[5]:

[3] apart from the Konishi operator

[4] N is even

[5] Here we use the conventions, terms and notations of ref. [22]

- $M|w$-strings: $M \geq 1$, $Y_{0|w} = 0$

$$\log Y_{M|w} = \log(1 + Y_{M-1|w})(1 + Y_{M+1|w}) \star s + \delta_{M1} \log \frac{1 - \frac{1}{Y_-}}{1 - \frac{1}{Y_+}} \,\hat{\star}\, s \,. \tag{2.1}$$

- $M|vw$-strings: $M \geq 1$, $Y_{0|vw} = 0$

$$\begin{aligned} \log Y_{M|vw}(v) = &-\delta_{M1} \sum_{j=1}^{N} \log S(u_j^- - v) - \sum_{M'=1}^{\infty} I_{M,M'} \sum_{j=1}^{n_{M'}} \log S(r_j^{(M')-} - v) \\ &+ \log(1 + Y_{M-1|vw})(1 + Y_{M+1|vw}) \star s + \delta_{M1} \log \frac{1 - Y_-}{1 - Y_+} \,\hat{\star}\, s - \log(1 + Y_{M+1}) \star s \,, \end{aligned} \tag{2.2}$$

with $I_{M,M'} = \delta_{M,M'-1} + \delta_{M,M'+1}$ and $n_M = 2(N-2)$ if $M \geq 1$, $n_0 = 0$. The first term is due to the pole of Y_+ at $u = u_j^- = u_j - \frac{i}{g}$, and the second term is due to the zeros of $1 + Y_{M|vw}$ at $u = r_j^{(M)-} = r_j^{(M)} - \frac{i}{g}$ which are subject to the quantization conditions:

$$\log Y_{M|vw}(r_j^{(M)-}) = 2\pi\, i\, I_j^{(M)}, \qquad r_j^{(M)} \in \mathbb{R}, \tag{2.3}$$

where the $I_j^{(M)}$s are half-integer quantum numbers.

- y-particles:

$$\log \frac{Y_+}{Y_-}(v) \;=\; -\sum_{j=1}^{N} \log S_{1_* y}(u_j, v) + \log(1 + Y_Q) \star K_{Qy} \,, \tag{2.4}$$

$$\begin{aligned} \log Y_+ Y_-(v) \;=\; &-\sum_{j=1}^{N} \log \frac{\left(S_{xv}^{1_*1}\right)^2}{S_2} \star s(u_j, v) - 2 \sum_{j=1}^{n_1} \log S(r_j^{(1)-} - v) \\ &+\; 2 \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s - \log\left(1 + Y_Q\right) \star K_Q + 2 \log(1 + Y_Q) \star K_{xv}^{Q1} \star s \,, \end{aligned} \tag{2.5}$$

where the second term in the second line is due to the zeros of $1 + Y_{1|vw}$ at $u = r_j^{(1)-}$.

- Q-particles for $Q \geq 2$

$$\log Y_Q \;=\; -2 \sum_{j=1}^{n_{Q-1}} \log S(r_j^{(Q-1)-} - v) + \log \frac{\left(1 + \frac{1}{Y_{Q-1|vw}}\right)^2}{(1 + \frac{1}{Y_{Q-1}})(1 + \frac{1}{Y_{Q+1}})} \star_{p.v.} s \,. \tag{2.6}$$

where the source term on the right hand side comes from the zeroes of $1 + Y_{Q-1|vw}$ at $u = r_j^{(Q-1)-}$. We note that the p.v. prescription is only necessary for $Q = 2$. For the $Q = 1$ case the hybrid version [22] of the TBA equations is more useful.

• Hybrid equation for the $Q = 1$ particle

$$\begin{aligned}\log Y_1(v) = &-\sum_{j=1}^{N}(\log S_{\mathfrak{sl}(2)}^{1_*1}(u_j, v) - 2\log S \star K_{vwx}^{11}(u_j^-, v))\\ &- L\,\tilde{\mathcal{E}}_1 + \log\big(1+Y_{Q'}\big) \star (K_{\mathfrak{sl}(2)}^{Q'1} + 2\,s \star K_{vwx}^{Q'-1,1})\\ &- 2\sum_{j=1}^{n_1}(\log S \,\hat{\star}\, K_{y1})\,(r_j^{(1)} - \frac{i}{g}, v) + 2\log(1+Y_{1|vw}) \star s \,\hat{\star}\, K_{y1}\\ &- 2\log\frac{1-Y_-}{1-Y_+} \,\hat{\star}\, s \star K_{vwx}^{11} + \log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}} \,\hat{\star}\, K_1 + \log\big(1-\frac{1}{Y_-}\big)\big(1-\frac{1}{Y_+}\big) \,\hat{\star}\, K_{y1}\,,\end{aligned} \tag{2.7}$$

where the first term on the right hand side comes from the zeros of $1 + Y_1$ at the magnon rapidities u_j while the source term in the third line of (2.7) is due to the zeros of $1 + Y_{1|vw}$ at $u = r_j^{(1)-}$. Here $L = 2 + J$ with $J = 2$ for the twist-two operators. We believe that this relation between the charge J and length L is valid for all states in the $sl(2)$ sector with a symmetric distribution of magnon rapidities. These include the minimal energy twist-two states discussed in this paper and all $N = 2$ states studied in [22].

The analytical continuation of (2.7) yields the exact Bethe equations for the real rapidities:

$$\begin{aligned}\pi i(2n_k+1) = \log Y_{1_*}(u_k) = &\; iL\,p_k - \sum_{j=1}^{N}\log S_{\mathfrak{sl}(2)}^{1_*1_*}(u_j, u_k)\\ &+ 2\sum_{j=1}^{N}\log \mathrm{Res}(S) \star K_{vwx}^{11_*}(u_j^-, u_k) - 2\sum_{j=1}^{N}\log\big(u_j - u_k - \frac{2i}{g}\big)\,\frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+}\\ &- 2\sum_{j=1}^{n_1}\Big(\log S \,\hat{\star}\, K_{y1_*}(r_j^{(1)-}, u_k) - \log S(r_j^{(1)} - u_k)\Big)\\ &+ \log\left(1+Y_Q\right) \star \Big(K_{\mathfrak{sl}(2)}^{Q1_*} + 2\,s \star K_{vwx}^{Q-1,1_*}\Big) + 2\log\big(1+Y_{1|vw}\big) \star (s \,\hat{\star}\, K_{y1_*} + \tilde{s})\\ &- 2\log\frac{1-Y_-}{1-Y_+} \,\hat{\star}\, s \star K_{vwx}^{11_*} + \log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}} \,\hat{\star}\, K_1 + \log\big(1-\frac{1}{Y_-}\big)\big(1-\frac{1}{Y_+}\big) \,\hat{\star}\, K_{y1_*}\,.\end{aligned} \tag{2.8}$$

The source and kernel functions together with the definition of the convolutions $\star$ and $\hat{\star}$ appearing in (2.1)-(2.8) can be found in [22].

3. The linearized problem

The $O(g^8)$ wrapping correction[6] to the ABA can be expressed by perturbing the TBA equations (2.1-2.8) around the asymptotic solution [25]. Borrowing the notation from [25]

[6] In the case of the twist-two ($J = 2$) operators the fact that the wrapping corrections start only at $O(g^8)$ can also be understood using superconformal invariance [2, 28].

for any Y function let Y^o be its asymptotic expression and $\mathscr{Y}$ the exponentially small perturbation around Y^o defined by the equation:

$$Y = Y^o(1+\mathscr{Y}). \tag{3.1}$$

Similarly we define the perturbation of the zeroes of $1+Y_{M|vw}$ by the formula:

$$r_j^{(M)} = \hat{r}_j^{(M)} + \delta r_j^{(M)}, \tag{3.2}$$

with $\hat{r}_j^{(M)}$ being the asymptotic value and $\delta r_j^{(M)}$ the small perturbation. Strictly speaking $\mathscr{Y}_{M|vw}$, as defined by (3.1) cannot be small everywhere, since the zeroes of the asymptotic solution are shifted by the perturbation and therefore $\mathscr{Y}_{M|vw}$ appears to have poles on the real axis. This problem can be avoided if we shift the integration contour for the $M|vw$ equations by (a small amount) $i\gamma$ below the real line before linearization. It is easy to see that the shifted contour does not cross any singularities and in the formulas (3.4) and (4.1) below this shift is understood (but not indicated explicitly). The actual calculation is performed in appendix A where the shifted contour is used throughout.

Expanding the TBA equations around the asymptotic values one finds the following set of linear equations for the perturbations:

- $M|w$-strings: $M \geq 1$, $\mathscr{Y}_{0|w} = 0$

$$\mathscr{Y}_{M|w} = (A_{M-1|w}\mathscr{Y}_{M-1|w} + A_{M+1|w}\mathscr{Y}_{M+1|w}) \star s + \delta_{M1}\left(\frac{\mathscr{Y}_+}{1-Y_+^o} - \frac{\mathscr{Y}_-}{1-Y_-^o}\right)\hat{\star}\, s\,, \tag{3.3}$$

where $A_{M|w} = \frac{Y^o_{M|w}}{1+Y^o_{M|w}}$.

- $M|vw$-strings: $M \geq 1$, $\mathscr{Y}_{0|vw} = 0$

$$\begin{aligned}\mathscr{Y}_{M|vw} &= (A_{M-1|vw}\mathscr{Y}_{M-1|vw} + A_{M+1|vw}\mathscr{Y}_{M+1|vw}) \star s - Y^o_{M+1} \star s \\ &-2\pi i \sum_{M'=1}^{\infty} I_{M,M'} \sum_{j=1}^{n_{M'}} s(\hat{r}_j^{(M')} - \frac{i}{g} - v)\, \delta r_j^{(M')} + \delta_{M1}\left(\frac{\mathscr{Y}_-}{1-\frac{1}{Y_-^o}} - \frac{\mathscr{Y}_+}{1-\frac{1}{Y_+^o}}\right)\hat{\star}\, s\,,\end{aligned} \tag{3.4}$$

with $A_{M|vw} \equiv \frac{Y^o_{M|vw}}{1+Y^o_{M|vw}}$. There are additional linear equations for the perturbations of the $r_j^{(M)}$s:

$$(\log Y^o_{M|vw})'(\hat{r}_j^{(M)} - \frac{i}{g})\, \delta r_j^{(M)} + \mathscr{Y}_{M|vw}(\hat{r}_j^{(M)} - \frac{i}{g}) = 0, \qquad M \geq 1, \qquad j = 1, ..., n_M \tag{3.5}$$

where the prime means differentiation with respect to the argument.

- y-particles

$$\mathscr{Y}_+ - \mathscr{Y}_- = Y^o_Q \star K_{Qy}\,, \tag{3.6}$$

$$\begin{aligned}\mathscr{Y}_+ + \mathscr{Y}_- &= 2(A_{1|vw}\mathscr{Y}_{1|vw} - A_{1|w}\mathscr{Y}_{1|w}) \star s - 4\pi i \sum_{j=1}^{n_1} s(\hat{r}_j^{(1)} - \frac{i}{g} - v)\, \delta r_j^{(1)} \\ &-Y^o_Q \star s + 2Y^o_Q \star K^{Q1}_{xv} \star s\,.\end{aligned} \tag{3.7}$$

Now we are in a position to analyze the magnitudes of the different terms with respect to the coupling g. For this purpose we further expand eqs. (3.3)-(3.7) with respect to the coupling. The source terms of the linear equations for the perturbations are given by convolution terms of the $Y_Q^o(u)$ functions. The $Y_Q^o(\frac{u}{g})$ functions can be generated using the results of ref. [27], and it turns out that similarly to the case of the Konishi field they are of $O(g^8)$ for small g [9]. Using this fact in can be shown that the convolution terms of the linear problem containing the $Y_Q^o(u)$ functions are at least of order g^8. This implies that also the perturbations are at least of $O(g^8)$. As $Y_Q^o \star K_{xv}^{Q1} \star s = O(g^8)$, from (3.7) it follows that $\mathscr{Y}_\pm = O(g^8)$, while (3.6) implies that $\mathscr{Y}_+ - \mathscr{Y}_- = O(g^9)$ since $Y_Q^o \star K_{Qy} = O(g^9)$.

Further from the asymptotic solution of the Y-functions we see that Y_+^o and Y_-^o coincide at leading order in g:

$$\frac{Y_+^o(u)}{Y_-^o(u)} = 1 + O(g^2).$$

This implies according to (3.3) that $\mathscr{Y}_{M|w}(\frac{u}{g}) = O(g^9)$ and that the order g^8 perturbations of the $Y_{M|vw}$s in (3.4),(3.5) are unaffected by the contributions of the $Y_\pm$ functions and decouple from the other type of variables.

4. The Bajnok-Janik formula

Finally we can turn our attention to the Bethe equations (2.8). In (2.8) the $Y_\pm$ functions appear only through $\hat{\star}$ type convolution terms this is why their perturbations give only $O(g^9)$ contribution. Consequently up to the order of g^8 only the perturbations of the $Y_{M|vw}$ functions contribute. Rescaling the variables $u \to u/g$, $u_k \to u_k/g$, $\hat{r}_j^{(m)} \to \xi_{m;j}/g$ and $\delta r_j^{(m)} \to \delta\xi_{m;j}/g$ and making similar considerations as in [25] and [26] it can be shown that the Bethe equations (2.8) up to $O(g^8)$ can be expressed by the leading $O(g^8)$ expressions of the Y_Q^o functions and with the solution of a linear problem coming from (3.4). This linear problem is similar to the linearization of the TBA equations of the XXX Heisenberg chain [26] and in the rescaled variables it takes the form:

$$\frac{\delta y_m}{y_m} - s \star (\delta L_{m-1} + \delta L_{m+1}) + \sum_{m'=1}^{\infty} I_{m,m'} \sum_{j=1}^{n_{m'}} g(u - \xi_{m';j})\delta\xi_{m';j} = -s \star Y_{m+1}^o, \quad m = 1, 2, ..., \tag{4.1}$$

$$\left(\frac{\delta y_m}{y_m}\right)(\xi_{m;j} - i) - y_m'(\xi_{m;j} - i)\,\delta\xi_{m;j} = 0, \quad m = 1, 2, ... \quad j = 1, ..., n_m, \tag{4.2}$$

where $g(u) = \frac{\pi}{2\sinh\frac{\pi}{2}u}$ and from now on[7] in the rest of the paper $s(u) = \frac{1}{4\cosh\frac{\pi}{2}u}$. $\delta L_m = (\delta y_m)/(1 + y_m)$ is the order g^8 perturbation of $\log(1 + Y_{m|vw})$ in the rescaled variables and $y_m(u) = \lim_{g\to 0} Y^o(\frac{u}{g})$. For the twist-two operators the asymptotic form of y_m corresponds to the $Q(u) = u$ solution of a compact site-N spin $\frac{1}{2}$ $(s = \frac{1}{2})$ XXX Heisenberg chain with the N inhomogeneity parameters given by the solutions of the Bethe equations of a site-2 non-compact spin minus $\frac{1}{2}$ $(s = -\frac{1}{2})$ Heisenberg chain.

[7]So far we have used the notations of [22], where $s(u) = \frac{g}{4\cosh\frac{\pi}{2}gu}$.

In terms of the solution of (4.1),(4.2) the wrapping corrected Bethe equations take the form:

$$\pi(2n_k+1) \,=\, J\,p_k + i\sum_{j=1}^{N}\log S_{\mathfrak{sl}(2)}^{1_*1_*}(\frac{u_j}{g},\frac{u_k}{g}) + \delta\mathcal{R}_k \,+ O(g^9), \tag{4.3}$$

with $\delta\mathcal{R}_k$ given by the formula

$$\delta\mathcal{R}_k = \delta\mathcal{R}_k^{(1)} + \delta\mathcal{R}_k^{(2)} + \delta\mathcal{R}_k^{(3)}, \tag{4.4}$$

where $\delta\mathcal{R}_k^{(1)}$ and $\delta\mathcal{R}_k^{(3)}$ comes from the small g expansion of the convolution terms containing the Y_Q functions in (2.8), while $\delta\mathcal{R}_k^{(2)}$ originates from the perturbation of the third line and the convolution terms containing the $Y_{1|vw}$ function in (2.8). Their explicit form is given by:

$$\delta\mathcal{R}_k^{(1)} = \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_m^o(u)\,\frac{u-u_k}{(m+1)^2+(u-u_k)^2}\,, \tag{4.5}$$

$$\delta\mathcal{R}_k^{(2)} = \int_{-\infty}^{\infty}\mathrm{d}u\,\frac{\delta L_1(u)}{2\sinh\frac{\pi}{2}(u-u_k)} + \sum_j\frac{\pi\,\delta\xi_{1;j}}{\cosh\frac{\pi}{2}(u_k-\xi_{1;j})} \tag{4.6}$$

and

$$\delta\mathcal{R}_k^{(3)} = \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^o(u)\,\left\{\mathcal{F}_m(u-u_k) - \frac{u-u_k}{m^2+(u-u_k)^2}\right\}, \tag{4.7}$$

where

$$\mathcal{F}_m(u) = \frac{-i}{4}\left\{\psi\left(\frac{m+iu}{4}\right) - \psi\left(\frac{m-iu}{4}\right) - \psi\left(\frac{m+2+iu}{4}\right) + \psi\left(\frac{m+2-iu}{4}\right)\right\} \tag{4.8}$$

with the usual ψ function $\psi(z) = \Gamma'(z)/\Gamma(z)$ and no principal value prescription is needed in (4.6) since the integrand is regular at $u = u_k$.

The nontrivial part of the calculation is the evaluation of $\delta\mathcal{R}_k^{(2)}$. The details of this calculation are given in appendix A. The result is[8]

$$\begin{aligned}\delta\mathcal{R}_k^{(2)} &= \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^o(u)\left\{\partial_k\log\hat{t}_m(u)\right\}\\ &= \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^o(u)\left\{\partial_k\log t_m(u) - \frac{r_m'(u-u_k)}{r_m(u-u_k)}\right\}.\end{aligned} \tag{4.9}$$

Using

$$\frac{r_m'(x)}{r_m(x)} = \mathcal{F}_m(x) - \frac{2x}{m^2+x^2} \tag{4.10}$$

[8]Strictly speaking we should use here the functions shifted away from the real axis by $-i\gamma$ as in appendix A, but after the identification (4.14) we see that potential singularities along the real axis actually cancel and we correctly get (4.15).

we find that the transcendental parts cancel and the result can be given in terms of rational functions:

$$\delta\mathcal{R}_k = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_1^o(u)\,\frac{u-u_k}{4+(u-u_k)^2}$$
$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^o(u)\left\{\partial_k \log t_m(u) + \frac{u-u_k}{m^2+(u-u_k)^2} + \frac{u-u_k}{(m+2)^2+(u-u_k)^2}\right\}. \tag{4.11}$$

This can be compactly written

$$\delta\mathcal{R}_k = \frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_m^o(u)\,\partial_k \log j_m(u)\,, \tag{4.12}$$

where[9]

$$j_m(u) = \frac{16 C_2 S_1^2 g^8}{(u^2+m^2)^4}\,\frac{t_{m-1}^2(u)}{t_0(u-im-i)\,t_0(u-im+i)\,t_0(u+im-i)\,t_0(u+im+i)}\,. \tag{4.13}$$

So far we have not used the explicit form of $Y_m^o(u)$. The crucial observation is that

$$Y_m^o(u) = j_m(u)\,, \tag{4.14}$$

which can be verified using the asymptotic solutions given in [27]. Thus (4.12) finally becomes

$$\delta\mathcal{R}_k = \frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, \partial_k\, Y_m^o(u)\,, \tag{4.15}$$

which is the Bajnok-Janik formula (1.4).

5. Conclusions

In this paper we have shown that the wrapping corrected Bethe equations of the twist-two operators of the $sl(2)$ sector coming from the generalized Lüscher approach and the TBA equations coincide up to $O(g^8)$. Our considerations are rather general and are not very sensitive to the details of the state under consideration. The analysis of the small g behaviour of the terms in the TBA equations suggest that also in the general case, in the calculation of the order g^{2L} correction of the Bethe equations, the linear problem related to the $Y_{M|vw}$ functions decouple from the rest of the linearized equations and the leading order wrapping correction to the Bethe equations is always given by (4.4), although the positions of the singularities are different from those of the twist-two operators. Next it can be recognized that the derivation presented in appendix A is insensitive to the distribution of the singularities associated to the $Y_{M|vw}$ functions indicating that the leading order g^{2L} wrapping correction of the Bethe equations of any symmetric state (the set of magnon rapidities consists of $\{u_j, -u_j\}$ pairs) with R-charge J of the $sl(2)$ sector is given by (1.4), with the appropriate asymptotic Y_Q^o functions.

[9] The constants C_2 and S_1 are defined in appendix B.

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A. Calculation of $\delta\mathcal{R}_k^{(2)}$

As it was mentioned in section 4, the asymptotic solution for the Y-system components associated to vw-strings (in the $g \to 0$ limit) is identical to a solution of the rational XXX-model with inhomogeneities. After suitable rescaling of the independent variable, and introducing the notation $Y^o_{m|vw} \to y_m$, we can recognize that these functions satisfy the standard form of the semi-infinite chain of XXX-model Y-system equations:

$$y_m(u+i)\,y_m(u-i) = [1+y_{m+1}(u)]\,[1+y_{m-1}(u)], \qquad m=1,2,\dots, \tag{A.1}$$

where $y_0(u)=0$ by convention. The XXX model, its Bethe Ansatz solution together with the corresponding T-system, Y-system, and Baxter TQ-relations, are of course very well known (for a review, see [29]). Here we summarize those elements of the solution that we need in our calculation.

A solution of the XXX Y-system relations (A.1) is given in terms of the solution of the corresponding T-system equations:

$$t_m(u+i)\,t_m(u-i) = t_{m+1}(u)\,t_{m-1}(u) + t_0(u+(m+1)i)\,t_0(u-(m+1)i), \tag{A.2}$$

where all $t_m(u)$ $(m=0,1,\dots)$ are polynomials starting with

$$t_0(u) = \prod_{j=1}^{N}(u-v_j). \tag{A.3}$$

Here the set of inhomogeneities $\{v_1,\dots,v_N\}$ can be fixed arbitrarily. For the AdS/CFT case we are interested in this set has the special form $\{u_1,-u_1,\dots,u_{N/2},-u_{N/2}\}$, but we will make this specialization only at the very end of the calculation. Starting from a given t_0, by solving the Bethe Ansatz equations, we can build our t_m functions. All these functions are polynomials of degree N. The twist-two states we are studying in this paper correspond to solutions with a single Bethe root (which is at zero for the symmetric magnon distribution $\{u_j,-u_j\}$). These are one of the simplest Bethe Ansatz solutions, but their explicit form is not needed in this calculation. The Y-system functions are given by the well-known formulas

$$y_m(u) = \frac{t_{m+1}(u)\,t_{m-1}(u)}{t_0(u+(m+1)i)\,t_0(u-(m+1)i)} \qquad m=1,2,\dots \tag{A.4}$$

and

$$1+y_m(u) = \frac{t_m(u+i)\,t_m(u-i)}{t_0(u+(m+1)i)\,t_0(u-(m+1)i)} \qquad m=0,1,\dots \tag{A.5}$$

The position of the roots and poles of y_m and $1+y_m$ are determined by the roots of the polynomials t_m using these formulas. Actually, in the TBA framework only those roots

that are within the "physical strip", i.e. which are real or their distance to the real axis is smaller than unity, play any role. We denote by $\{\xi_{m;j}\}_{j=1}^{n_m}$ the set of such roots of t_m. We define the sign[10] of a root by

$$\begin{aligned} \omega_{m;j} &= +1 \qquad \text{if} \qquad 0 \le \operatorname{Im} \xi_{m;j} < 1, \\ \omega_{m;j} &= -1 \qquad \text{if} \qquad -1 < \operatorname{Im} \xi_{m;j} < 0 \end{aligned} \tag{A.6}$$

and further define

$$\tilde{\xi}_{m;j} = \xi_{m;j} - i\omega_{m;j}. \tag{A.7}$$

We note that $n_0 = N$ and $\xi_{0;j} = v_j$, $j = 1, \dots, N$. Before proceeding, let us define the T-system elements in a new gauge by

$$\hat{t}_m(u) = \left\{ \prod_{j=1}^{N} r_m(u - v_j) \right\} t_m(u)\,, \tag{A.8}$$

where

$$r_m(u) = \frac{1}{4} \frac{\gamma(2+m+iu)\,\gamma(2+m-iu)}{\gamma(4+m+iu)\,\gamma(4+m-iu)} \tag{A.9}$$

with $\gamma(u) = \Gamma(u/4)$. It is easy to verify that (A.4-A.5) are simplified to

$$y_m(u) = \hat{t}_{m+1}(u)\,\hat{t}_{m-1}(u) \qquad m = 1, 2, \dots, \tag{A.10}$$

$$1 + y_m(u) = \hat{t}_m(u+i)\,\hat{t}_m(u-i) \qquad m = 0, 1, \dots \tag{A.11}$$

and that t_m and $\hat{t}_m$ have the same physical roots (and neither have poles).

To avoid any singularities, as explained in the main text we shift the real line by a small amount $-i\gamma$ in the negative imaginary direction. This means that the new physical strip becomes

$$-1 - \gamma < \operatorname{Im} u < 1 - \gamma\,, \tag{A.12}$$

which explains why real roots are classified here as positive. (We have to choose γ small enough so that no physical root is lost or no new physical root is created by this shift. This is possible if there are no roots on the boundary of the original physical strip.)

It is now standard to translate the functional relation (A.11) into a TBA type integral equation[11]

$$\hat{t}_m^{-\gamma}(u) = \tau_m(u)\, \exp\left\{(s \star L_m)(u)\right\}\,, \tag{A.13}$$

where $L_m(u) = \log(1 + y_m^{-\gamma}(u))$ and

$$\tau_m(u) = \prod_{j=1}^{n_m} \tanh \frac{\pi}{4}(u - i\gamma - \xi_{m;j}). \tag{A.14}$$

[10]Since later we will shift the integration contour by $-i\gamma$, real roots are counted as positive. An equivalent way of proceeding would have been to shift by $+i\gamma$, in which case the real roots would be classified negative.

[11]Using the notation $f^{-\gamma}(u) = f(u - i\gamma)$ for any function f.

Similarly the Y-system equations can be transformed to the TBA equations

$$y_m^{-\gamma}(u) = \tau_{m+1}(u)\,\tau_{m-1}(u)\,\exp\left\{(s\star[L_{m+1}+L_{m-1}])(u)\right\}, \qquad m=1,2,\dots \tag{A.15}$$

These are supplemented by the quantization conditions

$$y_m(\tilde{\xi}_{m;j}) = -1\,, \qquad m=1,2,\dots, \qquad j=1,\dots,n_m\,, \tag{A.16}$$

which follow from (A.11).

We now "linearize" the TBA equations (A.15) by taking their logarithmic derivative with respect to one of the inhomogeneity parameters, v_k:

$$\partial_k\,\ell_m = -H_{m+1} - H_{m-1} + s\star(\partial_k L_{m+1} + \partial_k L_{m-1})\,, \qquad m=1,2,\dots, \tag{A.17}$$

where

$$\partial_k = \frac{\partial}{\partial v_k}\,, \qquad \ell_m(u) = \log y_m^{-\gamma}(u), \qquad H_m(u) = \sum_{j=1}^{n_m} Q_{m;j}(u)\,\partial_k\xi_{m;j} \tag{A.18}$$

with

$$Q_{m;j}(u) = g(u - i\gamma - \xi_{m;j})\,, \qquad g(u) = \frac{\pi}{2\sinh\frac{\pi}{2}u}\,. \tag{A.19}$$

Similarly the linearized form of the quantization conditions is

$$\partial_k\ell_m(\tilde{\xi}_{m;j}+i\gamma) - y_m'(\tilde{\xi}_{m;j})\,\partial_k\xi_{m;j} = 0\,, \qquad m=1,2,\dots, \qquad j=1,\dots,n_m\,. \tag{A.20}$$

The above linearized problem is very similar to (4.1)-(4.2), obtained by the linearization of the full AdS/CFT TBA system in the main text (around the same XXX-model Bethe Ansatz solution). Rewriting those equations using the definitions introduced above and shifting the contour by $-i\gamma$ we get

$$\delta\,\ell_m = -h_{m+1} - h_{m-1} + s\star(\delta L_{m+1} + \delta L_{m-1}) + i_m\,, \qquad m=1,2,\dots, \tag{A.21}$$

where

$$h_m(u) = \sum_{j=1}^{n_m} Q_{m;j}(u)\,\delta\xi_{m;j}\,, \qquad i_m = -s\star X_m\,, \qquad X_m(u) = Y^o_{m+1}(u-i\gamma)\,. \tag{A.22}$$

We note that by convention $h_0 = \delta L_0 = 0$ here. After the shift the linearized quantization conditions become

$$\delta\ell_m(\tilde{\xi}_{m;j}+i\gamma) - y_m'(\tilde{\xi}_{m;j})\,\delta\xi_{m;j} = 0\,, \qquad m=1,2,\dots, \qquad j=1,\dots,n_m\,. \tag{A.23}$$

Apart from the fact that the deviation from the given Bethe Ansatz solution is caused by changing one of the inhomogeneity parameters in the first case and coupling to other nodes of the AdS/CFT diagram in the second, the only difference between (A.17) and (A.21) is that in the former there are no source terms and $H_0(u) = g(u - i\gamma - v_k) \neq 0$. We

can, however, change our conventions by putting $H_0 = 0$ also in this case and compensating this by adding a source term

$$i_m(u) = -\delta_{m1}\, g(u - i\gamma - v_k)\,. \tag{A.24}$$

After these changes the two linear problems have identical structure.

Let us now write out this linear structure in some detail. Arranging the two types of unknowns as two (infinite component) column vectors

$$\delta L = \begin{pmatrix} \delta L_1(u) \\ \delta L_2(u) \\ \vdots \end{pmatrix} \qquad \delta\xi = \begin{pmatrix} \delta\xi_{1;j} \\ \delta\xi_{2;j} \\ \vdots \end{pmatrix} \tag{A.25}$$

we can then write the linearized TBA equations schematically as

$$M_{11}\,\delta L + M_{12}\,\delta\xi = I_1 \tag{A.26}$$

and the linearized quantization conditions as

$$M_{21}\,\delta L + M_{22}\,\delta\xi = I_2. \tag{A.27}$$

Here

$$I_1 = \begin{pmatrix} i_1(u) \\ i_2(u) \\ \vdots \end{pmatrix} \tag{A.28}$$

and, since for later convenience we multiply the quantization conditions by $2\pi i\omega_{m;j}$,

$$I_2 = \begin{pmatrix} -2\pi i\omega_{1;j}\, i_1(\tilde{\xi}_{1;j} + i\gamma) \\ -2\pi i\omega_{2;j}\, i_2(\tilde{\xi}_{2;j} + i\gamma) \\ \vdots \end{pmatrix}. \tag{A.29}$$

The operator matrices M_{11} etc. are as follows.

$$M_{11} = \begin{pmatrix} D_1 & -\sigma & 0 & 0 & \dots \\ -\sigma & D_2 & -\sigma & 0 & \dots \\ 0 & -\sigma & D_3 & -\sigma & \dots \\ 0 & 0 & -\sigma & D_4 & \dots \\ & & \vdots & & \end{pmatrix}, \tag{A.30}$$

where $D_m = 1 + 1/y_m$, $\sigma = s\star$,

$$M_{12} = \begin{pmatrix} 0 & V_2 & 0 & 0 & \dots \\ V_1 & 0 & V_3 & 0 & \dots \\ 0 & V_2 & 0 & V_4 & \dots \\ 0 & 0 & V_3 & 0 & \dots \\ & & \vdots & & \end{pmatrix}, \tag{A.31}$$

with $V_{m;j}(u) = Q_{m;j}(u)$,

$$M_{21} = \begin{pmatrix} 0 & V_1^T & 0 & 0 & \dots \\ V_2^T & 0 & V_2^T & 0 & \dots \\ 0 & V_3^T & 0 & V_3^T & \dots \\ 0 & 0 & V_4^T & 0 & \dots \\ & & \vdots & & \end{pmatrix} , \tag{A.32}$$

where ^{T} denotes transposition. Finally the matrix elements of M_{22} are given by the formula

$$M_{22mm';jj'} = -2\pi i \omega_{m;j}\, y'_m(\tilde{\xi}_{m;j})\, \delta_{mm'}\, \delta_{jj'} + (2\pi)^2\, s(\xi_{m;j} - \xi_{m';j'})\, (\delta_{m+1\,m'} + \delta_{m-1\,m'})\,. \tag{A.33}$$

The crucial observation is that

$$M_{11}^T = M_{11}\,, \qquad M_{12}^T = M_{21}\,, \qquad M_{22}^T = M_{22}\,, \tag{A.34}$$

and consequently the big operator matrix of the linear problem is symmetric:

$$\mathbf{M}^T = \mathbf{M}\,, \qquad \mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\,. \tag{A.35}$$

This means that, if the inverse operator $\mathbf{R}$ exists (which we assume) then it must also be symmetric:

$$\mathbf{R}^T = \mathbf{R}\,, \qquad \mathbf{R} = \mathbf{M}^{-1}\,. \tag{A.36}$$

Writing it as a hypermatrix

$$\mathbf{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.37}$$

this symmetry property, in terms of its components, reads:

$$A_{m,m'}(u,v) = A_{m',m}(v,u)\,, \quad B_{m,m';j}(u) = C_{m';j,m}(u)\,, \quad D_{m;j,m';j'} = D_{m';j',m;j} \tag{A.38}$$

Using the components of the inverse matrix and the source term (A.24), we can write

$$\begin{aligned} \partial_k L_m(u) &= \int_{-\infty}^{\infty} \mathrm{d}v\, g(v_k + i\gamma - v)\, A_{1,m}(v,u) - (2\pi)^2 \sum_j C_{1;j,m}(u)\, s(v_k - \xi_{1;j})\,, \\ \partial_k \xi_{m;j} &= \int_{-\infty}^{\infty} \mathrm{d}v\, g(v_k + i\gamma - v)\, B_{1,m;j}(v) - (2\pi)^2 \sum_{j'} s(v_k - \xi_{1;j'})\, D_{1;j',m;j}\,, \end{aligned} \tag{A.39}$$

where we already used the symmetry properties (A.38).

For the AdS/CFT case, the source terms are of the form (A.22) and the solution of the linear problem will depend linearly on the functions X_m. We define:

$$\begin{aligned} \delta L_m(u) &= \sum_{m'} \int_{-\infty}^{\infty} \mathrm{d}w\, \delta L_m^{(m')}(u,w)\, X_{m'}(w)\,, \\ \delta \xi_{m;j} &= \sum_{m'} \int_{-\infty}^{\infty} \mathrm{d}w\, \delta \xi_{m;j}^{(m')}(w)\, X_{m'}(w)\,. \end{aligned} \tag{A.40}$$

To calculate the Lüscher correction, we only need δL_1 and $\delta\xi_{1;j}$:

$$\delta\mathcal{R}_k^{(2)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}v\, g(v-i\gamma-v_k)\,\delta L_1(v) + 4\pi\sum_j s(v_k-\xi_{1;j})\delta\xi_{1;j} \tag{A.41}$$

and we can define

$$\delta\mathcal{R}_k^{(2)} = \sum_m \int_{-\infty}^{\infty} \mathrm{d}w\, \delta\mathcal{R}_k^{(2)(m)}(w) X_m(w)\,. \tag{A.42}$$

Using the inverse operator we can write

$$\begin{aligned} \delta L_1^{(m)}(u,w) &= -\int_{\infty}^{\infty} \mathrm{d}v\, A_{1,m}(u,v)\, s(v-w) + \sum_j B_{1,m;j}(u) Q_{m;j}(w)\,,\\ \delta\xi_{1;j}^{(m)}(w) &= -\int_{\infty}^{\infty} \mathrm{d}v\, C_{1;j,m}(v)\, s(v-w) + \sum_{j'} D_{1;j,m;j'} Q_{m;j'}(w)\,. \end{aligned} \tag{A.43}$$

Note that exactly the same matrix elements of the inverse operator $\mathbf{R}$ appear here as in (A.39). Substituting these formulas into (A.41) and using the relations (A.39) we get

$$\begin{aligned} \pi\,\delta\mathcal{R}_k^{(2)(m)}(w) &= \int_{\infty}^{\infty} \mathrm{d}v\, \partial_k L_m(v)\, s(v-w) - \sum_j \left(\partial_k \xi_{m;j}\right)\, Q_{m;j}(w)\\ &= \partial_k (s\star L_m)(w) + \partial_k \log \tau_m(w) = \partial_k \log \hat{t}_m^{-\gamma}(w)\,. \end{aligned} \tag{A.44}$$

Finally we can write

$$\begin{aligned} \delta\mathcal{R}_k^{(2)} &= \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}w\, X_m(w)\partial_k \log \hat{t}_m^{-\gamma}(w)\\ &= \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}w\, Y_{m+1}^o(w-i\gamma)\left\{-\frac{r_m'(w-i\gamma-v_k)}{r_m(w-i\gamma-v_k)} + \partial_k \log t_m^{-\gamma}(w)\right\}\,. \end{aligned} \tag{A.45}$$

We can shift the integration contour back to the real axis at the end of the calculation.

B. Simplification of the Bajnok-Janik formula

This appendix is based on the results of ref. [9]. Unfortunately our conventions are different from that of this paper[12]. Here we write all formulae in our conventions.

The Lüscher correction to the energy is given by

$$\Delta E = -\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}q\, \frac{g^4}{(q^2+Q^2)^2}\varepsilon_Q\,, \tag{B.1}$$

where

$$\varepsilon_Q = \mathrm{STr}\left\{S_{Q;1}(q,v_1)S_{Q;1}(q,v_2)\ldots S_{Q;1}(q,v_N)\right\}. \tag{B.2}$$

Here $v_1 = u_1$, $v_2 = -u_1$, etc. is the symmetric magnon configuration. We want to calculate ε_Q in the lowest nontrivial order.

[12] $g = 2g^{\mathrm{BJL}}$, $u_k = 2u_k^{\mathrm{BJL}}$

Similarly the correction to the Bethe-Yang equations is given by

$$\delta\mathcal{R}_1 = -\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}q\, \frac{g^4}{(q^2+Q^2)^2}\Phi_Q\,, \tag{B.3}$$

where

$$\Phi_Q = \mathrm{STr}\left\{S'_{Q;1}(q,v_1)S_{Q;1}(q,v_2)\dots S_{Q;1}(q,v_N)\right\}. \tag{B.4}$$

Here $'$ means derivative withe respect to the variable q and again, we are interested in the lowest non-trivial order in g^2.

In what follows we concentrate on the contributions coming from a fixed Q sector. $S_{Q;1}(q,u)$, the S-matrix in this sector is a product of a scalar factor $\sigma(q,u)$ and a tensor product of two identical matrix factors. These matrices can be diagonalized in a u-independent way and can be written as

$$G(q)\mathcal{D}(q,u)G^{-1}(q)\,, \tag{B.5}$$

where $\mathcal{D}$ is diagonal:

$$\mathcal{D}(q,u) = \langle S_j^\alpha(q,u)\rangle\,, \tag{B.6}$$

and the eigenvalues can be grouped in such a way that their expansion is of the form

$$S_j^\alpha(q,u) = K^\alpha(u)\,A_j(q,u)\,\left\{1+g^2\delta_j^\alpha(q,u)\right\} + \mathrm{O}(g^4)\,, \tag{B.7}$$

where $j=0,1,\dots,Q-1$, $\alpha=0,1,2,3$ with 0,2 bosons, 1,3 fermions and since

$$K^\alpha(u)K^\alpha(-u) = 1\,, \tag{B.8}$$

$K^\alpha(u)$ plays no role for our symmetric configuration.

The scalar factor is given explicitly by

$$\sigma(q,u) = \frac{(u+i)^2}{\{(q-u)^2+(Q+1)^2\}\{(q-u)^2+(Q-1)^2\}} \tag{B.9}$$

and

$$A_j(q,u) = q-u+i(1+2j-Q)\,. \tag{B.10}$$

Using these building blocks, we can write

$$\varepsilon_Q = \sigma_1\sigma_2\dots\sigma_N\,m^2\,, \tag{B.11}$$

where $\sigma_k = \sigma(q,v_k)$ and the matrix part is

$$m = \sum_{j,\alpha}(-1)^\alpha S_{1j}^\alpha S_{2j}^\alpha \dots S_{Nj}^\alpha\,, \tag{B.12}$$

where $S_{kj}^\alpha = S_j^\alpha(q,v_k)$.

The crucial observation of ref. [9] is that

$$\sum_\alpha(-1)^\alpha = 0\,, \qquad \sum_\alpha(-1)^\alpha\delta_j^\alpha(q,u) = ih(u)\,r_j(q)\,, \tag{B.13}$$

where

$$h(u) = \frac{2}{1+u^2}\,, \qquad r_j(q) = \frac{1}{q+i(2j-Q)} - \frac{1}{q+i(2+2j-Q)}\,. \tag{B.14}$$

This is why the leading term vanishes and the next term becomes simple:

$$m = iC_1 g^2 \sum_{j=0}^{Q-1} A_{1j}A_{2j}\ldots A_{Nj} r_j(q)\,, \tag{B.15}$$

where $A_{kj} = A_j(q, v_k)$ and

$$C_1 = \sum_{k=1}^{N} h(v_k) = 2\sum_{k=1}^{N/2} h(u_k) = 2S_1(N) = 2\sum_{i=1}^{N}\frac{1}{i}\,. \tag{B.16}$$

In what follows, C_1, together with a similar constant coming from the scalar factors,

$$C_2(N) = \prod_{k=1}^{N}(1+v_k^2) = \left(\frac{2^N (N!)^3}{(2N)!}\right)^2 \tag{B.17}$$

are treated as numerical constants[13], which take definite numerical value for our twist-two states.

Φ_Q can be built from the same building blocks:

$$\Phi_Q = \sigma_1'\sigma_2\ldots\sigma_N\, m^2 + 2\sigma_1\sigma_2\ldots\sigma_N\, m\tilde{m}\,, \tag{B.18}$$

where

$$\tilde{m} = iC_1 g^2 \sum_{j=0}^{Q-1} A_{1j}'A_{2j}\ldots A_{Nj} r_j(q) + ig^2 h(v_1)\sum_{j=0}^{Q-1} A_{1j}A_{2j}\ldots A_{Nj} r_j'(q)\,. \tag{B.19}$$

This is obtained from the derivative acting on the eigenvalues. A potential additional term, where the derivative is acting on the diagonalizing matrices is of the form

$$\mathrm{STr}\left\{\left[G^{-1}G', \mathcal{D}\right](\mathrm{diag})\right\} \tag{B.20}$$

and since the commutator term is off-diagonal and the rest diagonal, this vanishes. The second term in (B.19) can be dropped, because the summands are odd under the symmetry $j \to Q-1-j$, $q \to -q$ and thus do not contribute after summation and integration, multiplied by a q-even function.

All the terms that are left depend only on the difference $q-u$ (like in relativistic models) and therefore the derivative with respect to q can be replaced by the derivative with respect to the first rapidity u_1. This leads to the identity

$$\Phi_Q \approx -\frac{1}{2}\frac{\partial}{\partial u_1}\varepsilon_Q\,. \tag{B.21}$$

[13]The value (B.17) can be calculated using the generalized hypergeometric representation of the leading order particle rapidities [9].

$\approx$ here means that the equality holds only after integrating both sides, multiplied by even functions. To prove (B.21) we can use a little lemma stating that for f, g even functions,

$$\int_{-\infty}^{\infty} \mathrm{d}q f'(q-\alpha)f(q+\alpha)g(q) = -\frac{1}{2}\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} \mathrm{d}q f(q-\alpha)f(q+\alpha)g(q)\,. \tag{B.22}$$

This applies directly to the $\sigma_1'\sigma_2\ldots\sigma_N$ term and for the first term in (B.19) we can use it combined with the $j \to Q-1-j$ symmetry.

Let us compute ε_Q explicitly. We find

$$\varepsilon_Q = \frac{16g^4 C_2(N) S_1^2(N) t_{Q-1}^2(q)}{(q^2+Q^2)^2 t_0(q+i+iQ)t_0(q-i-iQ)t_0(q-i+iQ)t_0(q+i-iQ)}\,, \tag{B.23}$$

where

$$t_0(q) = \prod_{k=1}^{N}(q-v_k) \tag{B.24}$$

and $t_m(q)$ are the XXX model T-system elements corresponding to the inhomogeneities (B.24) and Baxter's Q function

$$Q^{\mathrm{Baxter}}(q) = q\,, \tag{B.25}$$

i.e. one Bethe root at zero.

Thus our final result is

$$\delta\mathcal{R}_1 = \frac{1}{4\pi}\frac{\partial}{\partial u_1}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}q\, Y_Q^o(q) \tag{B.26}$$

with

$$Y_Q^o(q) = \frac{16g^8 C_2(N) S_1^2(N) t_{Q-1}^2(q)}{(q^2+Q^2)^4 t_0(q+i+iQ)t_0(q-i-iQ)t_0(q-i+iQ)t_0(q+i-iQ)}\,. \tag{B.27}$$

It is not difficult to see that in (B.26) we can make the substitution

$$\frac{\partial}{\partial u_1} \to 2\frac{\partial}{\partial v_1} = 2\partial_1\,, \tag{B.28}$$

where the derivative is understood as follows. First $v_1, v_2, \ldots, v_N$ are treated as independent inhomogeneity parameters, and only after the integration the derivative with respect to v_1 was taken is the configuration restricted again to the symmetric one.

C. The Bajnok-Janik formula in relativistic models

The generalized Lüscher approach is also valid in relativistic integrable models, like the Sine-Gordon model, nonlinear σ-models and other related models. In this appendix we summarize the results for the O(2), O(3) and O(4) nonlinear σ-models and the SU(n) principal model. We have proven the formulas using the known TBA/NLIE description of these models. The details of the derivation will be published elsewhere.

In the simple cases corresponding to the O(n) σ-models for $n = 2, 3$ and 4 there is only one type of particles of mass μ and there are no bound states. For a state consisting of r particles of rapidities $\{\theta_1, \dots, \theta_r\}$ in a very large periodic box of length L the energy of the system is simply

$$E^{(0)} = \sum_{j=1}^{r} \mu \cosh \theta_j \tag{C.1}$$

and the particle rapidities are subject to the quantization conditions

$$QC_k^{(0)}(\theta_1, \dots, \theta_r) = \mathrm{e}^{i\mu L \sinh \theta_k} \, \mathrm{e}^{i\mathcal{R}_k} = -1 \,, \qquad k = 1, \dots, r \,, \tag{C.2}$$

where

$$\mathrm{e}^{i\mathcal{R}_k} = \sigma(\theta_k | \theta_1, \dots, \theta_r) \tag{C.3}$$

and $\sigma(\theta|\theta_1, \dots, \theta_r)$ is the eigenvalue of the transfer matrix (constructed from the unitary and crossing symmetric physical S-matrix) corresponding to the given state.

The energy expression that includes the exponentially small first correction to the energy is given by Lüscher's formula:

$$E^{(1)} = E^{(0)} + \delta E = E^{(0)} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \, \mu \cosh \theta \, \mathrm{e}^{-\mu L \cosh \theta} \, \sigma \left(\theta + \frac{i\pi}{2} \Big| \theta_1, \dots, \theta_r \right) . \tag{C.4}$$

At the same exponential order the quantization condition is modified to

$$QC_k^{(1)}(\theta_1, \dots, \theta_r) = QC_k^{(0)}(\theta_1, \dots, \theta_r) \, \{1 + i \, \delta\mathcal{R}_k\} = -1 \,, \qquad k = 1, \dots, r \,, \tag{C.5}$$

where

$$\delta\mathcal{R}_k(\theta_1, \dots, \theta_r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \, \mathrm{e}^{-\mu L \cosh \theta} \, \partial_k \, \sigma \left(\theta + \frac{i\pi}{2} \Big| \theta_1, \dots, \theta_r \right) \tag{C.6}$$

with $\partial_k = \partial / \partial \theta_k$.

For the SU(n) principal model the formulae are more complicated since in this model there are several types of particles, one corresponding to each fundamental representation [30]. They can be indexed from 1 to $n-1$, 1 corresponding to the defining (vector) representation and $n-1$ to their antiparticles. We denote the corresponding masses by μ_a and by $\sigma^a(\theta|\theta_1, \dots, \theta_r)$ the transfer matrix eigenvalues corresponding to the a^{th} fundamental representation in the auxiliary space. For simplicity we here consider states with all particles belonging to the $n-1$ (anti-vector) representation only. In this case the Lüscher correction to the energy is given by

$$\begin{aligned} E^{(1)} &= E^{(0)} + \delta E = \sum_{j=1}^{r} \mu_{n-1} \cosh \theta_j \\ &+ \frac{i}{2\pi} \sum_{a=1}^{n-1} \mu_a \int_{-\infty}^{\infty} \mathrm{d}\theta \, \sinh \left(\theta + \frac{i\pi}{n} \right) \, \mathrm{e}^{i\mu_a L \sinh\left(\theta + \frac{i\pi}{n}\right)} \, \sigma^a \left(\theta + \frac{i\pi}{n} \Big| \theta_1, \dots, \theta_r \right) . \end{aligned} \tag{C.7}$$

Similarly the quantization conditions for $k = 1, \dots, r$ can be written as

$$QC_k^{(1)} = QC_k^{(0)} \, \{1 + i \, \delta\mathcal{R}_k\} = \mathrm{e}^{i\mu_{n-1} L \sinh \theta_k} \, \sigma^{n-1}(\theta_k | \theta_1, \dots, \theta_r) \, \{1 + i \, \delta\mathcal{R}_k\} = -1 \,, \tag{C.8}$$

where

$$\delta\mathcal{R}_k(\theta_1,\ldots,\theta_r)=\frac{1}{2\pi}\sum_{a=1}^{n-1}\int_{-\infty}^{\infty}\mathrm{d}\theta\;\mathrm{e}^{i\mu_a L\sinh\left(\theta+\frac{i\pi}{n}\right)}\,\partial_k\,\sigma^a\left(\theta+\frac{i\pi}{n}\Big|\theta_1,\ldots,\theta_r\right)\,. \qquad \text{(C.9)}$$

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